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Solvability of Multipoint Boundary Value Problems at Resonance for Higher-Order Ordinary Differential Equations

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Abstract—Let $f : [0,1] \times R^n \longrightarrow R$ be a continuous function and $e \in L^1[0,1]$. Let β_j $(1 \le j \le m-2) \in R$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ be given. This paper is concerned with the existence of solutions for the following n$^{\text{th}}$-order multipoint boundary value problems at resonance case

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \dots, x^{(n-1)}(t)\right) + e(t), \qquad t \in (0,1),$$
$$x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \qquad x(1) = \sum_{j=1}^{m-2} \beta_j x(\eta_j),$$

and

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \dots, x^{(n-1)}(t)\right) + e(t), \qquad t \in (0,1),$$
$$x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, \qquad x'(1) = \sum_{j=1}^{m-2} \beta_j x'(\eta_j).$$

Some existence results are obtained by using the coincidence degree theory of Mawhin.

Keywords—Multipoint boundary value problem, Resonance, Coincidence degree theory.

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1. INTRODUCTION

In this paper, we consider the following n^{th}-order boundary value problem (BVP),

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \dots, x^{(n-1)}(t)\right) + e(t), \qquad t \in (0,1), \tag{1}$$

$$x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0, \qquad x(1) = \sum_{j=1}^{m-2} \beta_j x(\eta_j), \tag{2}$$

and

$$x^{(n)}(t) = f\left(t, x(t), x'(t), \dots, x^{(n-1)}(t)\right) + e(t), \qquad t \in (0,1), \tag{1}$$

$$x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0, \qquad x'(1)t = \sum_{j=1}^{m-2} \beta_j x'(\eta_j). \tag{3}$$

Where $f : [0,1] \times R^n \longrightarrow R$ is a continuous function, $e \in L^1[0,1]$, β_j $(j = 1, \dots, m-2) \in R$ and $0 < \eta_1 < \eta_2 < \dots < \eta_{m-2} < 1$.

We say that BVP (1),(2) or BVP (1),(3) is a problem at resonance, if the linear equation,

$$x^{(n)}(t) = 0, \qquad t \in (0,1),$$

with the boundary condition (2) or (3) has nontrivial solutions. Otherwise, we call them a problem at nonresonance.

In the present work, if $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-1} = 1$, then, BVP (1),(2) is at resonance, since equation $x^{(n)}(t) = 0$ with boundary condition (2) has nontrivial solutions $x = ct^{n-1}$, $c \in R$, $t \in [0,1]$. If $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-2} = 1$, then BVP (1),(3) is at resonance, since equation $x^{(n)}(t) = 0$ with boundary condition (3) has nontrivial solutions $x = ct^{n-1}$, $c \in R$, $t \in [0,1]$.

Feng [1,2], Prezeradzki and Stanczy [3], Liu [4], and Gupta [5–7] studied the existence of solutions for some second-order multipoint boundary value problems at resonance. Gupta [8], Ma [9], and Nagle and Pothoven [10] studied the existence for some third-order third-point boundary value problems at resonance. However, rare works are done for higher-order multipoint boundary value problems at resonance.

For the nonresonance case of higher-order boundary value problems, for instance, we refer to [11,12] and the references therein.

The purpose of our paper is to discuss the existence of solutions for higher order multipoint BVP (1),(2) at resonance (i.e., $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-1} = 1$) and BVP (1),(3) at resonance (i.e., $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-2} = 1$). Our method is based on the coincidence degree theory of Mawhin [13,14].

2. MAIN RESULTS

For the convenience of the reader to understand the coincidence degree theory, we briefly recall some definitions [13–15].

DEFINITION 1. *Let Y, Z be real Banach spaces, the linear operator $L : \mathrm{dom}L \subset Y \longrightarrow Z$ is said to be a Fredholm map of index zero provided that* $\mathrm{Ker}L$, *the kernel of L, is of the same finite dimension as the* $Z/\mathrm{Im}L$, *where* $\mathrm{Im}L$ *is the image of L.*

Let L be a Fredholm map of index zero, and $P : Y \longrightarrow Y$, $Q : Z \longrightarrow Z$ be continuous projectors, such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L$ *and* $Y = \mathrm{Ker}L \oplus \mathrm{Ker}P$, $Z = \mathrm{Im}L \oplus \mathrm{Im}Q$. *We denote the inverse of the map* $L|_{\mathrm{dom}L \cap \mathrm{Ker}P} : \mathrm{dom}L \cap \mathrm{Ker}P \longrightarrow \mathrm{Im}L$ *by* K_P, i.e., $K_P = (L|_{\mathrm{dom}L \cap \mathrm{Ker}P})^{-1} : \mathrm{Im}L \longrightarrow \mathrm{dom}L \cap \mathrm{Ker}P$.

DEFINITION 2. *Let L be a Fredholm map of index zero and Ω be an open bounded subset of Y, such that* $\mathrm{dom}L \cap \Omega \neq \emptyset$*, the map $N : Y \longrightarrow Z$ is said to be L-compact on $\bar{\Omega}$, if the map $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \longrightarrow Y$ is compact.*

For more details, see [14,15].

The following theorem is the Mawhin theorem, which can be found in [13, Theorem IV.13] or [14, Theorem 2.4].

THEOREM A. *Let L be a Fredholm operator of index zero and let N be L-compact on $\bar{\Omega}$. Assume that the following conditions are satisfied.*

(i) $Lx \neq \lambda Nx$, *for every* $(x, \lambda) \in [(\mathrm{dom}L \setminus \mathrm{Ker}L) \cap \partial\Omega] \times (0, 1)$.

(ii) $Nx \notin \mathrm{Im}L$, *for every* $x \in \mathrm{Ker}L \cap \partial\Omega$.

(iii) $\deg(JQN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0) \neq 0$, *where* $J : \mathrm{Im}Q \longrightarrow \mathrm{Ker}L$ *is a linear isomorphism,* $Q : Z \longrightarrow Z$ *is a projection as above with* $\mathrm{Im}L = \mathrm{Ker}Q$.

Then, the equation $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \bar{\Omega}$.

In the following, we shall use the classical spaces $C[0,1], C^1[0,1], \dots, C^{n-1}[0,1]$ and $L^1[0,1]$. For $x \in C^{n-1}[0,1]$, we use the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$ and $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \dots, \|x^{(n-1)}\|_\infty\}$, and denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. We will use the Sobolev space $W^{n,1}(0,1)$, which may be defined by

$$W^{n,1}(0,1) = \Big\{x : [0,1] \longrightarrow R \mid x, x', \dots, x^{(n-1)} \text{ are absolutely continuous on } [0,1] \text{ with } x^{(n)} \in L^1[0,1]\Big\}.$$

Let $Y = C^{n-1}[0,1]$, $Z = L^1[0,1]$, L is the linear operator from $\mathrm{dom}L \subset Y$ to Z with

$$\mathrm{dom}L = \left\{x \in W^{n,1}(0,1) : x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0, x(1) = \sum_{j=1}^{m-2} \beta_j x(\eta_j)\right\}$$

and $Lx = x^{(n)}$, $x \in \mathrm{dom}L$. We define $N : Y \longrightarrow Z$ by setting

$$Nx = f\left(t, x(t), x'(t), \dots, x^{(n-1)}(t)\right) + e(t), \qquad t \in (0,1).$$

Then, BVP (1),(2) can be written as $Lx = Nx$.

In order to apply Theorem A, in the following Lemma 1, we shall show that L is a Fredholm operator of index zero and construct a linear continuous projector operator Q satisfying Condition (iii) in Theorem A.

LEMMA 1. *If* $\sum_{j=1}^{m-2} \beta_j = 1$, $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-1} = 1$, $\sum_{j=1}^{m-2} \beta_j \eta_j^n \neq 1$, *then,*

(i) $\mathrm{Ker}L = \{x \in \mathrm{dom}L : x = ct^{n-1}, c \in R, t \in [0,1]\}$;

(ii) $\mathrm{Im}L = \{y \in Z : \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \dots d\tau_n = 0\}$;

(iii) $L : \mathrm{dom}L \subset Y \longrightarrow Z$ *is a Fredholm operator of index zero, and the linear continuous projector operator* $Q : Z \longrightarrow Z$ *can be defined as*

$$Qy = \frac{n!}{1 - \sum_{j=1}^{m-2} \beta_j \eta_j^n} \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_n.$$

(iv) *The linear operator* $K_P : \mathrm{Im}L \longrightarrow \mathrm{dom}L \cap \mathrm{Ker}P$ *can be written as*

$$K_P y = \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_n$$

(v) $\|K_P y\| \leq \|y\|_1$, *for all* $y \in \mathrm{Im}L$.

PROOF.

(i) For $\forall x \in \mathrm{Ker}L$, we have $x^{(n)} = 0$. Then, we obtain

$$x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1},$$

where $a_i (i = 0, 1, \ldots, n-1) \in R$. From $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$, we have $a_0 = a_1 = \cdots = a_{n-2} = 0$. Again, from $x(1) = \sum_{j=1}^{m-2} \beta_j x(\eta_j)$, one has $a_{n-1} = a_{n-1} \sum_{j=1}^{m-2} \beta_j \eta_j^{n-1}$. In view of $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-1} = 1$, we have that

$$\mathrm{Ker}L = \left\{ x \in \mathrm{dom}L : x = ct^{n-1},\ c \in R,\ t \in [0,1] \right\}.$$

(ii) We show that

$$\mathrm{Im}L = \left\{ y \in Z : \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_n = 0 \right\}. \tag{4}$$

Since the problem

$$x^{(n)} = y \tag{5}$$

has a solution $x(t)$ satisfied $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0,\ x(1) = \sum_{j=1}^{m-2} \beta_j x(\eta_j)$, if and only if

$$\sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \ldots d\tau_n = 0. \tag{6}$$

In fact, if (5) has solution $x(t)$ satisfied $x(0) = x'(0) = \ldots = x^{(n-2)}(0) = 0,\quad x(1) = \sum_{j=1}^{m-2} \beta_j x(\eta_j)$, then, from (5), we have

$$\begin{aligned} x(t) &= x(0) + \frac{1}{1!} x'(0) t + \cdots + \frac{1}{(n-1)!} x^{(n-1)}(0) t^{n-1} + \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_n \\ &= \frac{1}{(n-1)!} x^{(n-1)}(0) t^{n-1} + \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_n. \end{aligned}$$

Thus, we have

$$x(1) = \frac{1}{(n-1)!} x^{(n-1)}(0) + \int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_n,$$

$$\sum_{j=1}^{m-2} \beta_j x(\eta_j) = \frac{1}{(n-1)!} x^{(n-1)}(0) \left(\sum_{j=1}^{m-2} \beta_j \eta_j^{n-1} \right) + \sum_{j=1}^{m-2} \beta_j \int_0^{\eta_j} \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_n.$$

According to $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-1} = 1$, we obtain

$$\sum_{j=1}^{m-2} \beta_j x(\eta_j) = \frac{1}{(n-1)!} x^{(n-1)}(0) + \sum_{j=1}^{m-2} \beta_j \int_0^{\eta_j} \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_n.$$

From $x(1) = \sum_{j=1}^{m-2} \beta_j x(\eta_j)$, we have

$$\int_0^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_n = \sum_{j=1}^{m-2} \beta_j \int_0^{\eta_j} \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_n,$$

again, from $\sum_{j=1}^{m-2}\beta_j = 1$, one has

$$\sum_{j=1}^{m-2}\beta_j \int_{\eta_j}^{1}\int_{0}^{\tau_n}\cdots\int_{0}^{\tau_2} y(\tau_1)\,d\tau_1\ldots d\tau_n = 0.$$

On the other hand, if (6) holds, setting

$$x(t) = ct^{n-1} + \int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_2} y(\tau_1)\,d\tau_1\cdots d\tau_n,$$

where c is an arbitrary constant, then $x(t)$ is a solution of (5), and $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0,\ x(1) = \sum_{j=1}^{m-2}\beta_j x(\eta_j)$. Hence, (4) is valid.

(iii) For $y \in Z$, we take the projector Qy as

$$Qy = \frac{n!}{1-\sum_{j=1}^{m-2}\beta_j\eta_j^n}\sum_{j=1}^{m-2}\beta_j\int_{\eta_j}^{1}\int_0^{\tau_n}\cdots\int_0^{\tau_2} y(\tau_1)\,d\tau_1\cdots d\tau_n.$$

Let $y_1 = y - Qy$, we obtain

$$\begin{aligned}\sum_{j=1}^{m-2}\beta_j\int_{\eta_j}^{1}\int_0^{\tau_n}\cdots\int_0^{\tau_2} y_1(\tau_1)\,d\tau_1\cdots d\tau_n &= \sum_{j=1}^{m-2}\beta_j\int_{\eta_j}^{1}\int_0^{\tau_n}\cdots\int_0^{\tau_2} y(\tau_1)\,d\tau_1\cdots d\tau_n \\ &\quad - Qy\sum_{j=1}^{m-2}\beta_j\int_{\eta_j}^{1}\int_0^{\tau_n}\cdots\int_0^{\tau_2} d\tau_1\cdots d\tau_n \\ &= 0,\end{aligned}$$

then, $y_1 \in \mathrm{Im}L$. Hence, $Z = \mathrm{Im}L + R$, since $\mathrm{Im}L \cap R = \{0\}$, we have $Z = \mathrm{Im}L \oplus R$, thus,

$$\dim \mathrm{Ker}L = \dim R = \mathrm{co}\dim \mathrm{Im}L = 1.$$

Hence, L is a Fredholm operator of index zero.

(iv) Taking $P : Y \longrightarrow Y$ as follows,

$$Px = x^{(n-1)}(0)\,t^{n-1},$$

then, the generalized inverse $K_P : \mathrm{Im}L \longrightarrow \mathrm{dom}L \cap \mathrm{Ker}P$ of L can be written as

$$K_P y = \int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_2} y(\tau_1)\,d\tau_1\cdots d\tau_n.$$

In fact, for $y \in \mathrm{Im}L$, we have

$$(LK_P)\,y(t) = [(K_P y)(t)]^{(n)} = y(t),$$

and, for $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, we know

$$\begin{aligned}(K_P L)\,x(t) &= \int_0^t\int_0^{\tau_n}\cdots\int_0^{\tau_2} x^{(n)}(\tau_1)\,d\tau_1\ldots d\tau_n \\ &= x(t) - x(0) - \frac{1}{1!}x'(0)\,t - \cdots - \frac{1}{(n-1)!}x^{(n-1)}(0)\,t^{n-1},\end{aligned}$$

in view of $x \in \mathrm{dom}L \cap \mathrm{Ker}P$, $x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0$, and $Px = 0$, thus,

$$(K_P L)\,x(t) = x(t).$$

This shows that $K_P = (L|_{\mathrm{dom}L\cap\mathrm{Ker}P})^{-1}$.

(v) From the definition of K_p, we have

$$\|K_P y\|_\infty \le \int_0^1 \cdots \int_0^1 |y(\tau_1)|\, d\tau_1 \cdots d\tau_n = \|y\|_1,$$

and from

$$(K_P y)'(t) = \int_0^t \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \cdots d\tau_{n-1},$$

$$\vdots$$

$$(K_P y)^{(n-1)}(t) = \int_0^t y(\tau_1)\, d\tau_1,$$

we obtain

$$\left\|(K_P y)'\right\|_\infty \le \|y\|_1, \ldots, \left\|(K_P y)^{(n-1)}\right\|_\infty \le \|y\|_1,$$

then, $\|K_P y\| \le \|y\|_1$.

This completes the proof of Lemma 1. ∎

THEOREM 1. *Let $f : [0,1] \times R^n \longrightarrow R$ be a continuous function, assume that*

(H$_1$) *There exist functions $a_1(t), a_2(t), \ldots, a_n(t)$, $b(t)$, $r(t) \in L^1[0,1]$, and constant $\theta \in [0,1)$, such that, for all $(x_1, x_2, \ldots, x_n) \in R^n$, $t \in [0,1]$, satisfying one of the following inequalities:*

$$|f(t, x_1, x_2, \ldots, x_n)| \le \left(\sum_{i=1}^n a_i(t)|x_i|\right) + b(t)|x_n|^\theta + r(t), \tag{7}$$

$$|f(t, x_1, x_2, \ldots, x_n)| \le \left(\sum_{i=1}^n a_i(t)|x_i|\right) + b(t)|x_{n-1}|^\theta + r(t), \tag{8}$$

$$\vdots$$

$$|f(t, x_1, x_2, \ldots, x_n)| \le \left(\sum_{i=1}^n a_i(t)|x_i|\right) + b(t)|x_1|^\theta + r(t). \tag{9}$$

(H$_2$) *There exists a constant $M > 0$, such that, for $x \in \mathrm{dom}L$, if $|x^{(n-1)}(t)| > M$, for all $t \in [0,1]$, then,*

$$\sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} \left[f\left(\tau_1, x(\tau_1), x'(\tau_1), \ldots, x^{(n-1)}(\tau_1)\right) + e(\tau_1)\right] d\tau_1 \cdots d\tau_n \ne 0. \tag{10}$$

(H$_3$) *There exists a constant $M^* > 0$, such that, for $c \in R$, if $|c| > M^*$, then, either*

$$c \cdot \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} \left[f\left(\tau_1, c\tau_1^{n-1}, c(n-1)\tau_1^{n-2}, \ldots, c \cdot (n-1)!\right) + e(\tau_1)\right] d\tau_1 \cdots d\tau_n < 0, \tag{11}$$

or else

$$c \cdot \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} \left[f\left(\tau_1, c\tau_1^{n-1}, c(n-1)\tau_1^{n-2}, \ldots, c \cdot (n-1)!\right) + e(\tau_1)\right] d\tau_1 \cdots d\tau_n > 0. \tag{12}$$

Then, for every $e(t) \in L^1[0,1]$, the BVP (1),(2) with $\sum_{j=1}^{m-2} \beta_j = 1$, $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-1} = 1$, $\sum_{j=1}^{m-2} \beta_j\ \eta_j^n \neq 1$, has at least one solution in $C^{n-1}[0,1]$, provided that $\sum_{i=1}^n \|a_i\|_1 < 1/2$.

PROOF. We need to construct the set Ω satisfying all the conditions in Theorem A, which is separated into the following four steps.

STEP 1. Let

$$\Omega_1 = \{x \in \mathrm{dom}L \setminus \mathrm{Ker}L : Lx = \lambda Nx, \text{ for some } \lambda \in [0,1]\}.$$

Then, Ω_1 is bounded.

Suppose that $x \in \Omega_1$, $Lx = \lambda Nx$, thus, $\lambda \neq 0$, $Nx \in \mathrm{Im}L = \mathrm{Ker}Q$, hence,

$$\sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^{\tau_n} \cdots \int_0^{\tau_2} \left[f\left(\tau_1, x(\tau_1), x'(\tau_1), \ldots, x^{(n-1)}(\tau_1)\right) + e(\tau_1)\right] d\tau_1 \cdots d\tau_n = 0,$$

thus, from (H_2), there exists $t_0 \in [0,1]$, such that $|x^{(n-1)}(t_0)| \leq M$. In view of

$$x^{(n-1)}(0) = x^{(n-1)}(t_0) - \int_0^{t_0} x^{(n)}(t)\, dt,$$

then,

$$\|Px\| = \left|x^{(n-1)}(0)\right| \leq M + \left\|x^{(n)}\right\|_1 = M + \|Lx\|_1 \leq M + \|Nx\|_1. \tag{13}$$

Again for $x \in \Omega_1, x \in \mathrm{dom}L\ \mathrm{Ker}L$, then $(I-P)x \in \mathrm{dom}L \cap \mathrm{Ker}P$, $LPx = 0$, thus from Lemma 1, we know

$$\|(I-P)x\| = \|K_P L(I-P)x\| \leq \|L(I-P)x\|_1 = \|Lx\|_1 \leq \|Nx\|_1. \tag{14}$$

From (13) and (14), we have

$$\|x\| \leq \|Px\| + \|(I-P)x\| = \left|x^{(n-1)}(0)\right| + \|(I-P)x\| \leq 2\|Nx\|_1 + M. \tag{15}$$

If (7) holds, then from (15), we obtain

$$\|x\| \leq 2\left[\left(\sum_{i=1}^n \|a_i\|_1 \left\|x^{(i-1)}\right\|_\infty\right) + \|b\|_1 \left\|x^{(n-1)}\right\|_\infty^\theta + C_1\right], \tag{16}$$

where $C_1 = \|r\|_1 + \|e\|_1 + M/2$. Thus, from $\|x\|_\infty \leq \|x\|$ and (16), we have

$$\|x\|_\infty \leq \frac{2}{1 - 2\|a_1\|_1}\left[\left(\sum_{i=2}^n \|a_i\|_1 \left\|x^{(i-1)}\right\|_\infty\right) + \|b\|_1 \left\|x^{(n-1)}\right\|_\infty^\theta + C_1\right]. \tag{17}$$

From $\|x'\|_\infty \leq \|x\|$, (16) and (17), one has

$$\begin{aligned}
\|x'\|_\infty &\leq \|x\| \\
&\leq 2\left[1 + \frac{2\|a_1\|_1}{1-2\|a_1\|_1}\right]\left[\left(\sum_{i=2}^n \|a_i\|_1 \left\|x^{(i-1)}\right\|_\infty\right) + \|b\|_1 \left\|x^{(n-1)}\right\|_\infty^\theta + C_1\right] \\
&= \frac{2}{1-2\|a_1\|_1}\left[\left(\sum_{i=2}^n \|a_i\|_1 \left\|x^{(i-1)}\right\|_\infty\right) + \|b\|_1 \left\|x^{(n-1)}\right\|_\infty^\theta + C_1\right],
\end{aligned}$$

then,

$$\begin{aligned}
\|x'\|_\infty &\leq \frac{2}{1 - 2(\|a_1\|_1 + \|a_2\|_1)}\left[\left(\sum_{i=3}^n \|a_i\|_1 \left\|x^{(i-1)}\right\|_\infty\right) + \|b\|_1 \left\|x^{(n-1)}\right\|_\infty^\theta + C_1\right], \\
&\vdots \\
\left\|x^{(n-1)}\right\|_\infty &\leq \frac{2}{1 - 2\sum_{i=1}^{n-1} \|a_i\|_1}\left[\|a_n\|_1 \left\|x^{(n-1)}\right\|_\infty + \|b\|_1 \left\|x^{(n-1)}\right\|_\infty^\theta + C_1\right],
\end{aligned} \tag{18}$$

then,

$$\left\|x^{(n-1)}\right\|_{\infty} \leq \frac{2\|b\|_1}{1-2\sum_{i=1}^{n}\|a_i\|_1}\left\|x^{(n-1)}\right\|_{\infty}^{\theta} + \frac{2C_1}{1-2\sum_{i=1}^{n}\|a_i\|_1}. \tag{19}$$

Since $\theta \in [0,1)$, from (19), there exist $M_1 > 0$, such that

$$\left\|x^{(n-1)}\right\|_{\infty} \leq M_1, \tag{20}$$

thus, from (20) and (18), there exist $M_2 > 0$, such that

$$\left\|x^{(n-2)}\right\|_{\infty} \leq M_2. \tag{21}$$

Similarly, there exist $M_i > 0, (i = 3, 4, \dots, n)$, such that

$$\left\|x^{(n-i)}\right\|_{\infty} \leq M_i, \qquad i = 3, 4, \dots, n, \tag{22}$$

hence,

$$\|x\| = \max\left\{\|x\|_{\infty}, \|x'\|_{\infty}, \dots, \left\|x^{(n-1)}\right\|_{\infty}\right\} \leq \max\{M_1, M_2, \dots, M_n\}.$$

Again, from (7), and (20)–(22), we have

$$\left\|x^{(n)}\right\|_1 \leq \|a_1\|_1 M_n + \cdots + \|a_{n-1}\|_1 M_2 + (\|a_n\|_1 + \|b\|_1) M_1 + \|r\|_1 + \|e\|_1.$$

Then, we show that Ω_1 is bounded.

If (8) or (9) holds, similar to the above argument, we can prove Ω_1 is bounded too.

STEP 2. The set $\Omega_2 = \{x \in \mathrm{Ker}L : Nx \in \mathrm{Im}L\}$ is bounded.

Let $x \in \Omega_2$, $x \in \mathrm{Ker}L = \{x \in \mathrm{dom}L : x = ct^{n-1}, c \in R, t \in [0,1]\}$, and $QNx = 0$, thus,

$$\sum_{j=1}^{m-2}\beta_j \int_{\eta_j}^{1}\int_0^{\tau_n}\cdots\int_0^{\tau_2}\left[f\left(\tau_1, c\tau_1^{n-1}, c(n-1)\tau_1^{n-2}, \dots, c\cdot(n-1)!\right)+e(\tau_1)\right] d\tau_1\cdots d\tau_n 0,$$

From (H_2), there exists $t_0 \in [0,1]$, such that $|x^{(n-1)}(t_0)| \leq M$, i.e., $|(n-1)!c| \leq M$. Then, we have $\|x^{(n-1)}\|_{\infty} = |x^{(n-1)}| = |(n-1)!c| \leq M$, thus,

$$\|x\| = \max\left\{\|x\|_{\infty}, \dots, \left\|x^{(n-1)}\right\|_{\infty}\right\} = \left\|x^{(n-1)}\right\|_{\infty} \leq M,$$

so the set Ω_2 is bounded.

STEP 3. If the first part of (H_3) holds, that is, there exists $M^* > 0$, such that, for any $c \in R$, if $|c| > M^*$, then,

$$\begin{gathered} c\cdot C_2\sum_{j=1}^{m-2}\beta_j\int_{\eta_j}^{1}\int_0^{\tau_n}\cdots\int_0^{\tau_2} \\ \left[f\left(\tau_1, c\tau_1^{n-1}, c(n-1)\tau_1^{n-2}, \dots, c\cdot(n-1)!\right)+e(\tau_1)\right] d\tau_1\cdots d\tau_n < 0, \end{gathered} \tag{23}$$

where

$$C_2 = \frac{n!}{1-\sum_{j=1}^{m-2}\beta_j\eta_j^n}.$$

Let

$$\Omega_3 = \{x \in \mathrm{Ker}L : -\lambda x + (1-\lambda) JQNx = 0,\ \lambda \in [0,1]\},$$

here, $J : \mathrm{Im}Q \longrightarrow \mathrm{Ker}L$ is the linear isomorphism given by $J(c) = ct^{n-1}$, $\forall c \in R,\ t \in [0,1]$. Then, Ω_3 is bounded.

Since, for $x = c_0 t^{n-1} \in \Omega_3$, then, for $t \in [0,1]$, we obtain

$$\lambda c_0 t^{n-1} = C_3 t^{n-1} \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau_n} \cdots \int_0^{\tau_2}$$
$$\left[f\left(\tau_1, c_0\tau_1^{n-1}, c_0 (n-1)\tau_1^{n-2}, \ldots, c_0 \cdot (n-1)!\right) + e(\tau_1)\right] d\tau_1 \cdots d\tau_n,$$

where

$$C_3 = (1-\lambda) C_2 = \frac{(1-\lambda)\cdot n!}{1 - \sum\limits_{j=1}^{m-2} \beta_j \eta_j^n},$$

or equivalently,

$$\lambda c_0 = C_3 \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau_n} \cdots \int_0^{\tau_2}$$
$$\left[f\left(\tau_1, c_0\tau_1^{n-1}, c_0 (n-1)\tau_1^{n-2}, \ldots, c_0 \cdot (n-1)!\right) + e(\tau_1)\right] d\tau_1 \cdots d\tau_n.$$

If $\lambda = 1$, then, $c_0 = 0$. Otherwise, if $|c_0| > M^*$, in view of (23), one has

$$\lambda c_0^2 = c_0 \cdot C_3 \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau_n} \cdots \int_0^{\tau_2}$$
$$\left[f\left(\tau_1, c_0\tau_1^{n-1}, c_0 (n-1)\tau_1^{n-2}, \ldots, c_0 \cdot (n-1)!\right) + e(\tau_1)\right] d\tau_1 \ldots d\tau_n < 0,$$

which contradicts $\lambda c_0^2 \geq 0$. Thus, $\Omega_3 \subset \{x \in \mathrm{Ker}L : \|x\| \leq (n-1)! M^*\}$ is bounded.

STEP 4. If the second part of (H$_3$) holds, that is, there exists $M^* > 0$, such that, for any $c \in R$, if $|c| > M^*$, then,

$$c \cdot C_2 \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau_n} \cdots \int_0^{\tau_2} \left[f\left(\tau_1, c\tau_1^{n-1}, c(n-1)\tau_1^{n-2}, \ldots, c \cdot (n-1)!\right) + e(\tau_1)\right] d\tau_1 \ldots d\tau_n > 0.$$

Let $\Omega_3 = \{x \in \mathrm{Ker}L : \lambda x + (1-\lambda) JQNx = 0, \lambda \in [0,1]\}$, here, J as in Step 3. Similar to the argument in Step 3, we can verify Ω_3 is bounded.

Now, we shall prove that all the conditions of Theorem A are satisfied.

Let Ω be a bounded open subset of Y, such that $\bigcup_{i=1}^3 \bar{\Omega}_i \subset \Omega$. By the Ascoli-Arzela theorem, we can show that $K_P(I-Q)N : \bar{\Omega} \longrightarrow Y$ is compact, thus, N is L-compact on $\bar{\Omega}$. Then, by the above argument, we have

(i) $Lx \neq \lambda Nx$, for every $(x,\lambda) \in [(\mathrm{dom}L \backslash \mathrm{Ker}L) \cap \partial\Omega] \times (0,1)$;
(ii) $Nx \notin \mathrm{Im}L$, for every $x \in \mathrm{Ker}L \cap \partial\Omega$;
(iii) If the first part of (H$_3$) holds, then, we let

$$H(x,\lambda) = -\lambda x + (1-\lambda) JQNx.$$

According to the above argument, we know $H(x,\lambda) \neq 0$, for $x \in \mathrm{Ker}L \cap \partial\Omega$, by the homotopy property of degree, we get

$$\begin{aligned} \deg\left(JQN\,|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0\right) &= \deg\left(H(\cdot,0), \Omega \cap \mathrm{Ker}L, 0\right) \\ &= \deg\left(H(\cdot,1), \Omega \cap \mathrm{Ker}L, 0\right) \\ &= \deg\left(-I, \Omega \cap \mathrm{Ker}L, 0\right). \end{aligned}$$

According to definition of degree on a space which is isomorphic to $R^l, l < \infty$, and

$$\Omega \cap \mathrm{Ker}L = \{ct^{n-1} : |c| < d\}.$$

We have

$$\begin{aligned}\deg\left(-I, \Omega \cap \mathrm{Ker}L, 0\right) &= \deg\left(-J^{-1}IJ, J^{-1}\left(\Omega \cap \mathrm{Ker}L\right), J^{-1}\{0\}\right) \\ &= \deg\left(-I, (-d, d), 0\right) = -1 \neq 0.\end{aligned}$$

If the second part of condition (3) of Theorem 2.2 holds, let

$$H(x, \lambda) = \lambda x + (1 - \lambda) JQNx.$$

Similar to the above argument, we have

$$\deg\left(JQN\mid_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0\right) = \deg\left(I, \Omega \cap \mathrm{Ker}L, 0\right) = 1,$$

since $0 \in \Omega \cap \mathrm{Ker}L$.

Then, we have

$$\deg\left(JQN|_{\mathrm{Ker}L}, \Omega \cap \mathrm{Ker}L, 0\right) \neq 0.$$

Then by, Theorem A, $Lx = Nx$ has at least one solution in $\mathrm{dom}L \cap \bar{\Omega}$, so that the BVP (1),(2) has at least one solution in $C^{n-1}[0,1]$. The proof is completed. ∎

Now, we discuss the existence of solution of BVP (1),(3).

Let $Y = C^{n-1}[0,1]$, $Z = L^1[0,1]$, the map N and the linear operator L are the same as above, and let

$$\mathrm{dom}L = \left\{x \in W^{n,1}(0,1) : x(0) = x'(0) = \cdots = x^{(n-2)}(0) = 0,\ x'(1) = \sum_{j=1}^{m-2} \beta_j x'(\eta_j)\right\}.$$

By using the same method as in the proof of Lemma 1 and Theorem 1, we can show the following Lemma 2 and Theorem 2.

LEMMA 2. *If $\sum_{j=1}^{m-2} \beta_j = 1$, $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-2} = 1$, $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-1} \neq 1$, then,*

(i) $\mathrm{Ker}L = \{x \in \mathrm{dom}L : x = ct^{n-1}, c \in R, t \in [0,1]\}$;

(ii) $\mathrm{Im}L = \{y \in Z : \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \ldots d\tau_{n-1} = 0\}$;

(iii) *$L : \mathrm{dom}L \subset Y \longrightarrow Z$ is a Fredholm operator of index zero, and the linear continuous projector operator $Q : Z \longrightarrow Z$ can be defined as*

$$Qy = \frac{(n-1)!}{1 - \sum\limits_{j=1}^{m-2} \beta_j \eta_j^{n-1}} \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^1 \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \ldots d\tau_{n-1},$$

(iv) *the linear operator $K_P : \mathrm{Im}L \longrightarrow \mathrm{dom}L \cap \mathrm{Ker}P$ can be written by*

$$K_P y = \int_0^t \int_0^{\tau_n} \cdots \int_0^{\tau_2} y(\tau_1)\, d\tau_1 \ldots d\tau_n;$$

(v) $\|K_P y\| \leq \|y\|_1$, for all $y \in \mathrm{Im}L$.

THEOREM 2. *Let $f : [0,1] \times R^n \longrightarrow R$ be a continuous function, assume that* (H_1) *in Theorem 1 is satisfied, and*

(H_4) *There exists a constant $M > 0$, such that, for any $x \in \mathrm{dom}L$, if $|x^{(n-1)}(t)| > M$, for all $t \in [0,1]$, then,*

$$\sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} \left[f\left(\tau_1, x(\tau_1), x'(\tau_1), \dots, x^{(n-1)}(\tau_1)\right) + e(\tau_1) \right] d\tau_1 \dots d\tau_{n-1} \neq 0.$$

(H_5) *There exists a constant $M^* > 0$, such that, for $c \in R$, if $|c| > M^*$, then, either*

$$c \cdot \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} \left[f\left(\tau_1, c\tau_1^{n-1}, c(n-1)\tau_1^{n-2}, \dots, c \cdot (n-1)!\right) + e(\tau_1) \right] d\tau_1 \dots d\tau_{n-1} < 0,$$

or else

$$c \cdot \sum_{j=1}^{m-2} \beta_j \int_{\eta_j}^{1} \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} \left[f\left(\tau_1, c\tau_1^{n-1}, c(n-1)\tau_1^{n-2}, \dots, c \cdot (n-1)!\right) + e(\tau_1) \right] d\tau_1 \dots d\tau_{n-1} > 0.$$

Then, for every $e(t) \in L^1[0,1]$, the BVP (1),(3) with $\sum_{j=1}^{m-2} \beta_j = 1$, $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-2} = 1$, $\sum_{j=1}^{m-2} \beta_j \eta_j^{n-1} \neq 1$, has at least one solution in $C^{n-1}[0,1]$, provided that $\sum_{i=1}^{n} \|a_i\|_1 < 1/2$.

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